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A tentative approach to entanglement measures for a system of a three-level atom interacting with a quantized cavity-field

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Abstract. In this paper, an entanglement measure due to quasi-mutual entropy from initially entangled mixed states of a three-level atom interacting with a single cavity field is introduced. Detailed analytical and explicit expressions are given taking into account an arbitrary form of the intensity-dependent coupling. Despite its simplicity the model exhibits a very broad range of intricate physical effects and it is widely used in quantized theories of laser. We show that quantum revivals are possible for a broad continuous range of physical parameters in the case of initial coherent states. Entanglement degree effects are shown to be very sensitive to the initial state of the system. Numerical calculations under current experimental conditions are taken into account and it is found that the intensity-dependent coupling changes the general features dramatically.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Hk Quantum communication – 42.65.-k Nonlinear optics – 32.80.Rm Multiphoton ionization and excitation to highly excited states (e.g., Rydberg states)

1 Introduction

With the recent rapid developments in quantum information there has been a renewed interest in multiparticle quantum mechanics and entanglement. The properties of states between the pure, maximally-entangled, and completely mixed limits are not completely known and are now under discussion [1–3]. Entanglement has occupied a central place in modern research because of its promise of enormous utility in quantum computing, quantum information, etc. A major thrust of current research is to find a quantitative measure of entanglement for general states. Approaches to this question based on the eigenvalue spectra of the system density matrices such as entropy methods, have given necessary but not sufficient conditions for particular states. Recently important questions have been raised [4] concerning the ability of entropy methods to decide on the question of separability of a composite state. There are a number of measures of entanglement for a bipartite system. The entanglement of formation, the entanglement of distillation [5], the concurrence [6] and relative entropy [7] are some of these measures. A method using quantum mutual entropy to measure the degree of entanglement in the time development of the Jaynes-Cummings model has been adopted in [8] and the case of the two-level atom with squeezed state has been studied [9]. The question of how mixed a two-level system and a field mode may be such that free entanglement arises in the course of the time evolution according to a Jaynes-Cummings type interaction has been considered [10,11].

It is rather interesting to note that entangled states play a crucial role in quantum computation as well, and it is the entanglement between qubits that gives a quantum computer its inherent advantage. However, works dealing with the entanglement in mixed states have been limited to the two-level systems [8–11]. It is therefore desirable to investigate the entanglement of the three-level systems. From the viewpoint of the Phoenix-Knight [12,13] entropy formalism, we have investigated the quantum field entropy and entanglement of a coherent field interacting with a three-level atom interacting with a single mode [14] and multimode [15]. However the method used in those papers cannot be applied when the system is taken to be initially in a mixed state.

In the present work we consider the situation for which the three-level system is initially in a mixed state. We essentially generalize the entanglement degree due to the quasi-mutual entropy, usually employed in the two-level system, to the three-level system interacting with a single cavity mode, including an arbitrary form of the intensity-dependent coupling. The physical situations which we shall refer to, belong to the experimental domains of cavity quantum electrodynamics. After the realization of a

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two-photon single mode maser operated in a high-Q cavity [16–18], a large amount of experimental and theoretical research has been done on the atomic two-photon transition processes in a cavity. Our work is organized as follows: in Section 2, we rewrite the dynamics of the three-level atom interacting with a cavity field from a point of view based on the entangled dressed-state eigenbasis and give exact expression for the unitary operator U_t. With the help of an appropriate coordinate system, in Section 3 we investigate the properties of the entanglement degree due to the quasi-mutual entropy. We devote Section 4 to give our discussion in which we assumed that the electromagnetic field is in coherent, and that the atom is initially in the mixed state. Finally, a summary of the main points of this work ends the paper and a few avenues for further investigations are indicated.

2 The system and its dynamics

The scheme we are going to discuss exploits the passage of a single atom only through the cavity [18]. We wish to underline the relevance of this aspect from an experimental point of view. Preparing and controlling a single atom is certainly much easier to achieve with respect to the case when the manipulation of many atoms is required. In addition, taking into consideration the low efficiency [19] of the atomic state detectors today used in laboratory, conditional measurement procedures involving one atom only instead of many ones, have to be preferred. The dynamics of several Hamiltonian models describing such systems is exactly treatable and, in most cases, testable in the laboratory. The point to be appreciated is indeed that, studying such systems, one has the opportunity to induce entanglement and to control its evolution in a multipartite physical system. In this paper, we consider the atomic system displayed in Figure 1. We study a three-level atom injected into a cavity field in a V-configuration, where the dipole-allowed transitions between the lower level $|a\rangle_A$ and the upper levels $|b\rangle_A$ and $|c\rangle_A$ are nonresonant with the cavity mode. The transition between the two upper levels is dipole forbidden. Furthermore, we assume the interaction including an arbitrary form of nonlinearity of the intensity-dependent coupling. In the rotating wave approximation, the interaction of the cavity mode with the injected atom is described by the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \omega_a \hat{S}_{aa} + \omega_b \hat{S}_{bb} + \omega_c \hat{S}_{cc} + \Omega \hat{a}^\dagger \hat{a} + \hat{\pi}\left(\gamma_1 \hat{S}_{ba} + \gamma_2 \hat{S}_{ca}\right) + \left(\gamma_1 \hat{S}_{ab} + \gamma_2 \hat{S}_{ac}\right)\hat{\pi}^\dagger. \quad (1)$$

Here, $\hat{\pi} = \hat{a} \otimes f(\hat{a}^\dagger \hat{a})$, where $\hat{a}$ and $\hat{a}^\dagger$, respectively, are the annihilation and the creation operators for the mode of the cavity field, and Ω the field frequency. We denote by $\gamma_i f(a^\dagger a)$ an arbitrary intensity-dependent coupling (see for example Refs. [20–22]). The operator $\widehat{S}_{ii}$ $(i = a, b, c)$ describes the atomic population of level $|i\rangle_A$ with energy ω_i and the operator $\widehat{S}_{ij}, (i \neq j)$ describes the transition from level $|i\rangle_A$ to level $|j\rangle_A$. The frequencies of transition

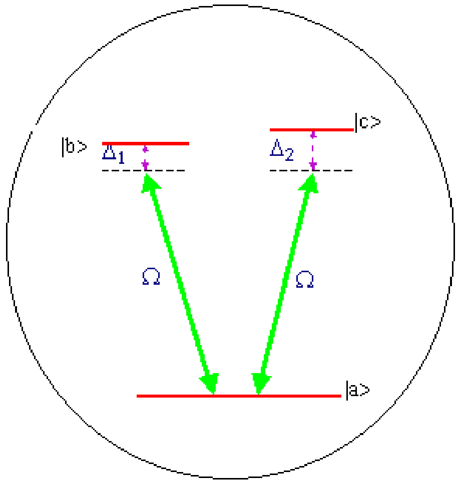


Fig. 1. The V-type three-level atom interacting with a single-mode field. The levels $|a\rangle_A, |b\rangle_A$, and $|c\rangle_A$ have the energy values $\hbar\omega_a$, $\hbar\omega_b$ and $\hbar\omega_c$, respectively. The transitions $|a\rangle_A \longrightarrow |b\rangle_A$, and $|a\rangle_A \longrightarrow |c\rangle_A$, are coupled to one and the same discrete intra-cavity mode $\hat{a}$ with an eigenfrequency Ω. The detunings of the levels $|a\rangle_A$, $|b\rangle_A$, and $|a\rangle_A$, $|c\rangle_A$, are $\Delta_1 = \omega_{ab} - \Omega$ and $\Delta_2 = \omega_{ac} - \Omega$, respectively.

between the levels $|a\rangle_A$ and $|b\rangle_A$ is ω_{ab}, and between the levels $|a\rangle_A$ and $|c\rangle_A$ is ω_{ac}. The parameters γ_i are corresponding atom-field coupling constants.

In general, mixed states are entangled if it is impossible to represent the density operator as an incoherent sum of factorizable pure states [23]. In pure-state quantum mechanics the state of the system is usually represented by a normalized wavefunction, which is a unit vector in a Hilbert space. If the system is in the pure state $|\psi(t)\rangle$ then $\rho(t)$ is simply the projector onto this state, *i.e.*, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, in such a way that $\rho^2(t) = \rho(t)$ and $\mathrm{Tr}\rho^2(t) = 1$. A mixed state instead is defined by the class of states which satisfy the inequality $\mathrm{Tr}\rho^2 < 1$. We assume that, before entering the cavity, the atom is prepared in a mixed state. To this end, the initial state of the atom can be written in the following form

$$\rho_A(0) = \varsigma_1 \widehat{S}_{aa} + \varsigma_2 \widehat{S}_{bb} + \varsigma_3 \widehat{S}_{cc} \in S_A, \quad (2)$$

where $\varsigma_i \geq 0$, and $\varsigma_1 + \varsigma_2 + \varsigma_3 = 1$. Also we suppose that the initial state of the field is given by

$$\rho_F(0) = |\varpi\rangle\langle\varpi| \in S_F, \quad (3)$$

where $|\varpi\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$, and $b_n^2 = |\langle\varpi|n\rangle|^2$ being the probability distribution of photon number for the initial state, with the normalization condition $\sum_{n=0}^{\infty} b_n^2 = 1$. The continuous map $\mathcal{E}_t^*$ describing the time evolution between the atom and the field is defined by the unitary operator generated by $\hat{H}$ such that

$$\mathcal{E}_t^*\colon S_A \longrightarrow S_A \otimes\ S_F,$$
$$\mathcal{E}_t^*\rho = \hat{U}_t\left(\rho_A(0)\otimes\rho_F(0)\right)\hat{U}_t^*, \tag{4}$$
$$\hat{U}_t \equiv \exp\left(-\mathrm{i}t\frac{\hat{H}}{\hbar}\right).$$

When we take the two-photon resonance condition $\Delta_1 = \Delta_2 = \Delta$, this unitary operator U_t can be written as

$$\hat{U}_t = \sum_{n=0}^{\infty}\left\{\exp\left(-\mathrm{i}tE_1^{(n)}\right)\left|\Psi_1^{(n)}\right\rangle\left\langle\Psi_1^{(n)}\right| + \exp\left(-\mathrm{i}tE_2^{(n)}\right)\right.$$
$$\left.\times\left|\Psi_2^{(n)}\right\rangle\left\langle\Psi_2^{(n)}\right| + \exp\left(-\mathrm{i}tE_3^{(n)}\right)\left|\Psi_3^{(n)}\right\rangle\left\langle\Psi_3^{(n)}\right|\right\}$$
$$+\exp\left(-\mathrm{i}tE^{(0)}\right)\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|, \tag{5}$$

where

$$E^{(0)} = \Delta,$$
$$E_j^{(n)} = (j-1)\,2^{(2-j)}\left[\delta + (-1)^j\mu_n\right], \quad j = 1,2,3 \tag{6}$$

are the eigenvalues with

$$\mu_n = \sqrt{\gamma_1^2 n f^2(n) + \gamma_2^2 n f^2(n) + \delta^2},$$

$\delta = \Delta/2$, and the detuning parameter Δ is defined as $\Delta = \omega_{ab} - \Omega = \omega_{ac} - \Omega$. The parameter $\mu(n)$ is a modified Rabi frequency. Hence we can easily express the eigenvectors of an atom in the cavity in the interaction picture in the form

$$|\Psi^{(0)}\rangle = |0\rangle \otimes |a\rangle_A\,,$$
$$|\Psi_1^{(n)}\rangle = C_{11}\Phi_2^{(n)} - C_{13}\Phi_3^{(n)},$$
$$|\Psi_2^{(n)}\rangle = C_{21}\Phi_1^{(n)} + C_{22}\Phi_2^{(n)} + C_{23}\Phi_3^{(n)},$$
$$|\Psi_3^{(n)}\rangle = C_{31}\Phi_1^{(n)} + C_{32}\Phi_2^{(n)} + C_{33}\Phi_3^{(n)}, \tag{7}$$

where

$$\begin{pmatrix}\Phi_1^{(n)}\\ \Phi_2^{(n)}\\ \Phi_3^{(n)}\end{pmatrix} = \begin{pmatrix}|n\rangle\otimes|a\rangle_A\\ |n-1\rangle\otimes|b\rangle_A\\ |n-1\rangle\otimes|c\rangle_A\end{pmatrix},$$

$$\mu_1 = \sqrt{\gamma_1^2 n f^2(n) + \gamma_2^2 n f^2(n)}$$
$$C_{11} = \frac{\gamma_2 f(n)\sqrt{n}}{\mu_1}, \qquad C_{13} = \frac{\gamma_1 f(n)\sqrt{n}}{\mu_1},$$
$$C_{21} = \frac{(\delta+\mu_n)}{\sqrt{2\mu_n^2 + \Delta\mu_n}}, \quad C_{31} = \frac{(\delta-\mu_n)}{\sqrt{2\mu_n^2 - \Delta\mu_n}},$$
$$C_{22} = \frac{\gamma_1 f(n)\sqrt{n}}{\sqrt{2\mu_n^2 + \Delta\mu_n}}, \quad C_{32} = \frac{\gamma_1 f(n)\sqrt{n}}{\sqrt{2\mu_n^2 - \Delta\mu_n}},$$
$$C_{23} = \frac{\gamma_2 f(n)\sqrt{n}}{\sqrt{2\mu_n^2 + \Delta\mu_n}}, \quad C_{33} = \frac{\gamma_2 f(n)\sqrt{n}}{\sqrt{2\mu_n^2 - \Delta\mu_n}}.$$

Having obtained the explicit form of the unitary operator U_t, the eigenvalues and the eigenfunctions for the system under consideration, we are therefore in a position to discuss the entanglement of the system.

It is important to point out that the increased insight into the dynamics of the three-level systems may be helpful in developing quantum information theory [24]. Recently, there is much interest in three-level quantum systems to represent information. It was demonstrated that key distributions based on three-level quantum systems are more secure against eavesdropping than those based on two-level systems [25–27]. Key distribution protocols based on entangled three-level systems were also proposed [28]. The security of these protocols is related to the violation of the Bell inequality. The three-level system provides in this context a much smaller level of noise [29,30]. Rydberg atoms which cross superconductive cavities are an almost ideal system to generate entangled states and to perform small scale quantum information processing [31]. In this context entanglement generation of three-level quantum systems was reported [24,32–35].

3 Derivation of the entanglement degree

Quantifying the amount of entanglement between quantum systems is a recent pursuit that has attracted a diverse range of researchers [5–15]. In this section, we will apply the results obtained in the previous section to derive the entanglement degree for a single three-level atom interacting with a cavity field without using the diagonal approximation method adapted in [8,9]. With a certain unitary operator, the final state after the interaction between the atom and the field is given by

$$\mathcal{E}_t^*\rho = U_t\left(\rho\otimes\varpi\right)U_t^*$$
$$= \varsigma_1 U_t|\varpi,\varpi;a\rangle\langle\varpi,\varpi;a|U_t^* + \varsigma_2 U_t|\varpi,\varpi;b\rangle$$
$$\times\langle\varpi,\varpi;b|U_t^* + \varsigma_3 U_t|\varpi,\varpi;c\rangle\langle\varpi,\varpi;c|U_t^*. \tag{8}$$

Therefore the von Neumann entropy of the total system is given by

$$S(\mathcal{E}_t^*\rho) = -\varsigma_1\log\varsigma_1 - \varsigma_2\log\varsigma_2 - \varsigma_3\log\varsigma_3. \tag{9}$$

Taking the partial trace over the atomic system, we obtain

$$\rho_t^F = \mathrm{Tr}_A\mathcal{E}_t^*\rho$$
$$= \varsigma_i\sum_{i=1}^{3}|\psi_i(t)\rangle\langle\psi_i(t)| + \varsigma_2\sum_{i=4}^{6}|\psi_i(t)\rangle\langle\psi_i(t)|$$
$$+\varsigma_3\sum_{i=7}^{9}|\psi_i(t)\rangle\langle\psi_i(t), \tag{10}$$

where

$$|\psi_1(t)\rangle = \sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t))\left(\cos\mu_n t + \mathrm{i}\delta\frac{\sin(\mu_n t)}{\mu_n}\right)|n+1\rangle,$$
$$|\psi_2(t)\rangle = -\mathrm{i}\sum_{n=0}^{\infty} b_n \exp(\mathrm{i}\delta t)\gamma_1 f(n)\sqrt{n}\frac{\sin(\mu_n t)}{\mu_n}|n\rangle,$$
$$|\psi_3(t)\rangle = -\mathrm{i}\sum_{n=0}^{\infty} b_n \exp(\mathrm{i}\delta t)\gamma_2 f(n)\sqrt{n}\frac{\sin(\mu_n t)}{\mu_n}|n\rangle, \quad (11)$$

$$|\psi_4(t)\rangle = -\mathrm{i}\sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t)\gamma_1 \times\sqrt{n}f(n)\sqrt{n}\frac{\sin(\mu_n t)}{\mu_n}|n+1\rangle$$
$$|\psi_5(t)\rangle = \sum_{n=0}^{\infty} b_n\left(1-\frac{n\gamma_1^2 f^2(n)}{\mu_n^2}+\frac{n\gamma_1^2 f^2(n)}{\mu_n^2}\exp(\mathrm{i}\delta t) \times\cos(\mu_n t)+\mathrm{i}\delta n\gamma_1^2 f^2(n)\exp(\mathrm{i}\delta t)\frac{\sin\mu_n t}{\mu_n^3}\right)|n\rangle,$$
$$|\psi_6(t)\rangle = \sum_{n=0}^{\infty} b_n\left(\frac{n\gamma_1\gamma_2 f^2(n)}{\mu_n^2}\cos(\mu_n t)+\mathrm{i}n\delta\gamma_1\gamma_2 f^2(n) \times\exp(\mathrm{i}\delta t)\frac{\sin(\mu_n t)}{\mu_n^3}-\frac{n\gamma_1\gamma_2 f^2(n)}{\mu_n^2}\right)|n\rangle, \quad (12)$$

$$|\psi_7(t)\rangle = -\mathrm{i}\sum_{n=0}^{\infty} b_n \exp(-\mathrm{i}\delta t)\gamma_2 \times\sqrt{n}f(n)\sqrt{n}\frac{\sin(\mu_n)}{\mu(n)}|n+1\rangle,$$
$$|\psi_8(t)\rangle = \sum_{n=0}^{\infty} b_n\left(\frac{n\gamma_1\gamma_2 f^2(n)}{\mu_n^2}\cos(\mu_n t)+\mathrm{i}n\delta\gamma_1\gamma_2 f^2(n) \times\exp(\mathrm{i}\delta t)\frac{\sin(\mu_n t)}{\mu_n^3}-\frac{n\gamma_1\gamma_2 f^2(n)}{\mu_n^2}\right)|n\rangle,$$
$$|\psi_9(t)\rangle = \sum_{n=0}^{\infty} b_n\left(1-\frac{n\gamma_2^2 f^2(n)}{\mu_n^2}+\frac{n\gamma_2^2 f^2(n)}{\mu_n^2}\exp(\mathrm{i}\delta t) \times\cos(\mu_n t)+\mathrm{i}\delta n\gamma_2^2 f^2(n)\exp(\mathrm{i}\delta t)\frac{\sin\mu_n t}{\mu_n^3}\right)|n\rangle. \quad (13)$$

Then the von Neumann entropy for the reduced state $S(\rho_t^F)$ is computed by

$$S(\rho_t^F) = -\sum_{i=1}^{9}\lambda_i^F(t)\log\lambda_i^F(t), \quad (14)$$

where $\{\lambda_i^F(t)\}$ are the solutions of

$$\det[\rho(\hat{t})-\lambda(\hat{t})N(\hat{t})]=0, \quad (15)$$

where $\rho(\hat{t})$ and $N(\hat{t})$ are 9×9 matrices having the following elements

$$\left[\rho(\hat{t})\right]_{ij} \equiv \langle\psi_i(t)|\rho_t^F|\psi_j(t)\rangle, \quad (i,j=1,2,3,...9),$$
$$\left[N(\hat{t})\right]_{ij} \equiv \langle\psi_i(t)|\psi_j(t)\rangle, \quad (i,j=1,2,3,...9). \quad (16)$$

On the other hand, the final state of the atomic system is given by taking partial trace over the field system:

$$\rho_t^A \equiv \mathrm{Tr}_F\mathcal{E}_t^*\rho \equiv \Im_1|a\rangle\langle a|+\Im_2|a\rangle\langle b|+\Im_3|a\rangle\langle c|+\Im_4|b\rangle\langle a|+\Im_5|b\rangle\langle b| +\Im_6|b\rangle\langle c|+\Im_7|c\rangle\langle a|+\Im_8|c\rangle\langle b|+\Im_9|c\rangle\langle c|, \quad (17)$$

where $\Im_i$ are given by

$$\Im_i = \sum_{k=1}^{3}\left\{Y_k C_{11}^{(n)}C_{ki}^{*(m)}+Y_{k+3}C_{21}^{(n)}C_{ki}^{*(m)} +Y_{k+6}C_{31}^{(n)}C_{ki}^{*(m)}\right\}, \quad i=1,2,3$$
$$\Im_j = \sum_{k=1}^{3}\left\{Y_k C_{12}^{(n)}C_{kj-3}^{*(m)}+Y_{k+3}C_{22}^{(n)}C_{kj-3}^{*(m)} +Y_{k+6}C_{32}^{(n)}C_{kj-3}^{*(m)}\right\}, \quad j=4,5,6$$
$$\Im_r = \sum_{k=1}^{3}\left\{Y_k C_{13}^{(n)}C_{kr-6}^{*(m)}+Y_{k+3}C_{23}^{(n)}C_{kr-6}^{*(m)} +Y_{k+6}C_{33}^{(n)}C_{kr-6}^{*(m)}\right\}, \quad r=7,8,9 \quad (18)$$

and, Y_i are given by

$$Y_i = \exp(-\mathrm{i}tE_{1i}^{(nm)})\{\varsigma_1 C_{12}^{*(n)}C_{i2}^{(m)}+\varsigma_2 C_{11}^{*(n)}C_{i1}^{(m)} +\varsigma_3 C_{13}^{*(n)}C_{i3}^{(m)}\}, \quad i=1,2,3$$
$$Y_j = \exp(-\mathrm{i}tE_{2j}^{(nm)})\{\varsigma_1 C_{22}^{*(n)}C_{j2}^{(m)}+\varsigma_2 C_{21}^{*(n)}C_{j1}^{(m)} +\varsigma_3 C_{23}^{*(n)}C_{j3}^{(m)}\}, \quad j=4,5,6$$
$$Y_r = \exp(-\mathrm{i}tE_{3r}^{(nm)})\{\varsigma_1 C_{32}^{*(n)}C_{r2}^{(m)}+\varsigma_2 C_{31}^{*(n)}C_{r1}^{(m)} +\varsigma_3 C_{33}^{*(n)}C_{r3}^{(m)}\}, \quad r=7,8,9 \quad (19)$$

where, $\exp(-\mathrm{i}tE_{ij}^{(nm)})=\exp(-\mathrm{i}t(E_i^{(n)}-E_j^{(m)})$.

Then the von Neumann entropy for the reduced state $S(\rho_t^A)$ is computed by

$$S(\rho_t^A) = -\sum_{i=1}^{3}\lambda_i^A(t)\log\lambda_i^A(t), \quad (20)$$

where $\lambda_i^A(t)$ is given by

$$\lambda_1^A(t) = -\frac{1}{3}-\frac{2}{3}\left(\sqrt{1-3\vartheta_1}\right)\cos(\beta),$$
$$\lambda_2^A(t) = -\frac{1}{3}+\frac{1}{3}(\cos(\beta)+\sqrt{3}\sin(\beta))\left(\sqrt{1-3\vartheta_1}\right),$$
$$\lambda_3^A(t) = -\frac{1}{3}+\frac{1}{3}(\cos(\beta)-\sqrt{3}\sin(\beta))\left(\sqrt{1-3\vartheta_1}\right), \quad (21)$$

where

$$\beta = \frac{1}{3}\cos^{-1}\left(\frac{2-9\vartheta_1-27\vartheta_2}{2(1-3\vartheta_1)^{3/2}}\right),$$
$$\vartheta_1 = \Im_1\Im_9+\Im_1\Im_5+\Im_5\Im_9-|\Im_6|^2-|\Im_2|^2-|\Im_3|^2,$$
$$\vartheta_2 = \Im_1\Im_5\Im_9+\Im_2\Im_7\Im_6-\Im_1|\Im_6|^2-\Im_9|\Im_2|^2-\Im_5|\Im_3|^2. \quad (22)$$

Using the above equations, the final expression for the entanglement degree in the three-level system takes the following form

$$\begin{aligned} I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right) &\equiv \mathrm{Tr}\mathcal{E}_t^*\rho(\log\mathcal{E}_t^*\rho-\log(\rho_t^A\otimes\rho_t^F)) \\ &= S(\rho_t^A)+S(\rho_t^F)-S(\mathcal{E}_t^*\rho) \\ &= -\sum_{i=1}^{9}\lambda_i^F(t)\log\lambda_i^F(t)-\sum_{i=1}^{3}\lambda_i^A(t)\log\lambda_i^A(t) \\ &\quad +\sum_{i=1}^{3}\varsigma_i\log\varsigma_i. \end{aligned} \quad (23)$$

It is evident that, with the help of equation (23), it is possible to study the entanglement degree of any three-level system when the system is initially in a mixed state.

Here, for simplicity, if we confine ourselves to V-type three-level atom interacting with a single cavity field and if $\mathcal{E}_t^*\rho\in S_1\otimes S_2$ is an entangled pure state, then its von Neumann entropy is equal to 0 ($S(\mathcal{E}_t^*\rho)=0$). Moreover, according to the triangle inequality of Araki and Lieb [36], we have $S(\rho_t^A)=S(\rho_t^F)$. Thus we find $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right)=2S(\rho_t^F)$. To be sure this entanglement measure can be computed, like any physical quantity in quantum mechanics, from knowledge of the density matrix which can be found experimentally with tomography [37,38], but their relation to experimental consequences are indirect at best. Equation (23) will be the basis of the numerical investigation. As one can see, it is unlikely to express the sums in the above equations in a closed form, however for reasonably large value of $\bar{n}$, direct numerical evaluations can be performed. Interesting features resulting from the different parameters are discussed in the following section.

An example of a truly mixed state for which the entanglement manipulations have been proven to be asymptotically reversible has been reported in reference [39]. It has been proven that the positivity of the partial transpose entanglement cost for the exact preparation of a large class of quantum states under positivity of the partial transpose operations is given by the logarithmic negativity [2], thus they have provided an operational meaning to the logarithmic negativity. Here we focus on the time development of the entangled state in a three-level system by applying entanglement degree due to quasi-mutual entropy [40] which is a special case of the quantum relative entropy type measure. It provides an upper bound on the entanglement of distillation and is more readily calculable than it.

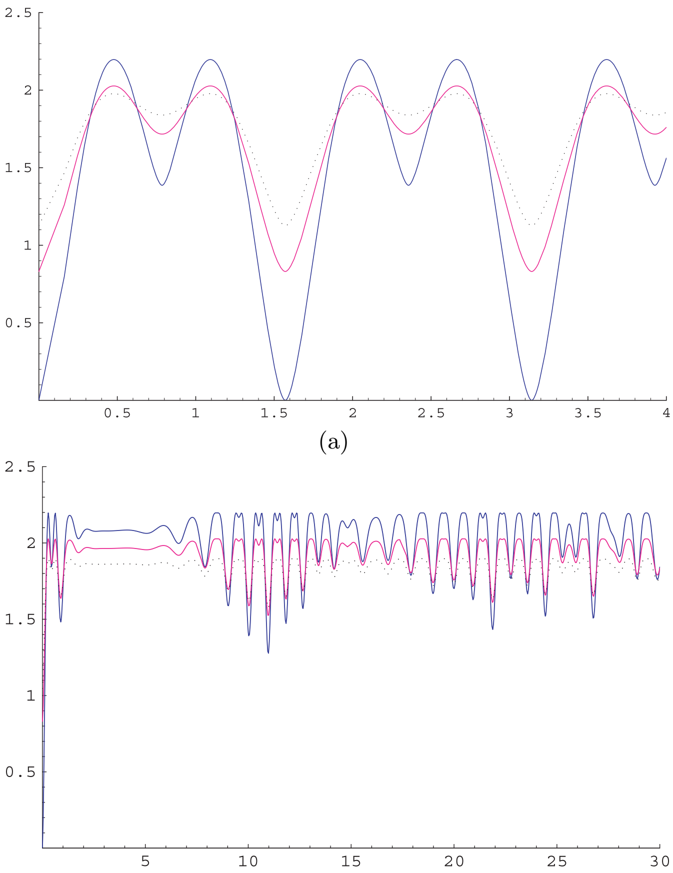
4 Numerical simulations

The material that is presented below demonstrates the mathematical soundness of our treatment. Besides that, it is of great use in performing a numerical evaluation of our entanglement degree expression equation (23). We shall be able to examine the influence of different parameters on the time evolution of the entanglement degree. At a special choice of the parameters ς_i such as $\varsigma_1=1$ ($\varsigma_2=1$), *i.e.*, the atom initially in the lower (upper) state, the final state of the system becomes a pure entangled state. Therefore it is sufficient to use von Neumann entropy in order to measure the degree of entanglement for the above cases. Then entanglement degree takes just twice the reduced von Neumann entropy *i.e.*,

$$\begin{aligned} I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right) &\equiv \mathrm{Tr}\mathcal{E}_t^*\rho(\log\mathcal{E}_t^*\rho-\log(\rho_t^A\otimes\rho_t^F)) \\ &= S(\rho_t^A)+S(\rho_t^F)-0; \\ &= 2S(\rho_t^A). \end{aligned} \quad (24)$$

These situations have been considered and the reduced von Neumann entropy has been applied to analyze the quantum fluctuations [12–15]. In a general case (*i.e.*, $\varsigma_1\neq 0$ or 1), the final state does not necessarily become a pure state, so that we need to adopt the $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right)$ in order to measure the degree of entanglement in the present model. Thus our initial setting enable us to discuss the variation of the entanglement degree for different values of the parameter ς_1 of the initial atomic system. Which means that we have presented a general framework for entanglement degree of a three-level quantum systems, and have shown that using quantum mutual entropy to calculate the entanglement degree open new possibilities for better understanding of multi-level atoms interactions when the atom initially in a mixed state. This allows us to study the entanglement degree of the system and convert from pure states into mixed states, which is crucial for many applications in quantum optics, physics and computing.

We will first compare the entanglement degree obtained here with that obtained when the atom initially prepared in a pure state in order to show the validity of our measure. In Figure 2 we show the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right)$as a function of the scaled time γt for a number state ($n=1$), where the physical parameters are taken from references [41,42]. In Figure 2a, we consider $\zeta_3=0$, and different values of the parameter $\zeta_1=0.001, 0.8$ and 0.6. We assume exact resonance case $\Delta=0$, and the nonlinear intensity-dependent coupling $f(n)=1$. In the case of $\zeta_1=0.001$, as time goes on we note a growth in the entanglement degree, followed by a sudden decrease, almost down to zero at $n\pi/2$. We find that the maximum value of the entanglement in this case is given by $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right)\approx 2.18$. When we further increase the parameter $\varsigma_1\approx 0$ we find that the degree of entanglement takes just twice value of the von Neumann entropy *i.e.*, $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A,\rho_t^F\right)\approx 2\log 3$. It is interesting to see here the periodic oscillations of the entanglement degree in this case. When we consider $\varsigma_1=0.8$, we see that the



(a)

(b)

Fig. 2. The evolution of the entanglement degree $I_{\mathcal{E}_t^* \rho}\left(\rho_t^A, \rho_t^F\right)$ as functions of the scaled time $\gamma_1 t$. The intensity-dependent atom-field coupling $f(n) = 1$, the detuning parameter Δ has zero value, $\varsigma_3 = 0$ and $\varsigma_1 = 0.001$ (sold curve), $\varsigma_1 = 0.8$ (dashed-curve) and $\varsigma_1 = 0.6$ (dotted-curve). (a) the field initially in a number state with $n = 1$ and (b) the field initially in a coherent state with $\bar{n} = 5$.

maximum value of the entanglement degree decreases (see Fig. 2a). Also, in this case we show that the amplitude of the oscillations is decreased also. The maximum values of the entanglement degree as well as the amplitude of the oscillations are decreased when we increase ς_1 further.

In Figure 2b we consider the same situation in Figure 2a but for a coherent state. In physical terms, b_n^2 corresponds to the probability distribution for measurements of the total excitation number operator on the initial state. In the particular case of a field initially in a coherent state, the probability distribution is given by $b_n^2 = \exp(-\bar{n})\bar{n}^n/n!$, where $\bar{n}$ is the mean photon number of photons present initially in the field. Let us mention that this distribution has its maximum around $\bar{n}$. It is remarked that, the first maximum of the entanglement degree at $\gamma_1 t > 0$ is achieved at the collapse time, and at one-half of the revival time, the entanglement degree reaches its local minimum. Meanwhile, the general feature of the entanglement degree in the case ς_1 takes values such as $\varsigma_1 \approx 0.8$ is also almost identical to that in the previous cases ($\zeta_1 = 0.001$) but the maximum value and the amplitude are decreased. It has been shown that the entanglement degree undergoes a collapse followed by a series of revivals (see Fig. 2b). When we consider $\varsigma_1 = 0.6$, we see that the maximum value of the entanglement degree decreases further. Also, in this case we show that the amplitude of the oscillations is decreased. The same result as in Figure 2a is observed here, as we further increase the parameter $\varsigma_1 \approx 0$ we find that $I_{\mathcal{E}_t^* \rho}\left(\rho_t^A, \rho_t^F\right)$ takes just twice value of the von Neumann entropy *i.e.*, $I_{\mathcal{E}_t^* \rho}\left(\rho_t^A, \rho_t^F\right) \approx 2\log 3$. To estimate the revival times of the Rabi oscillations in the limit of large one-photon detuning, we follow an analogous procedure given in reference [43]. We assume that the dominant contribution in the summation from the term for which $n \approx \bar{n}$, where $\bar{n}$, is the mean photon number for which the initial photon number distribution is maximum. Then the times of revivals t_R of the Rabi oscillations is given by $t_R \approx 2\pi\sqrt{2\bar{n}}$.

Now we will turn our attention to the effect on the entanglement degree of the nonlinearity of the intensity-dependent coupling as an example, *i.e.* $f(n) = \sqrt{n}$, in Figure 3a, and $f(n) = 1/\sqrt{n}$ in Figure 3b. Comparing the behavior in Figure 3a, with cases considered in Figure 3b, we may say that the effect of the intensity coupling is rather different, where the oscillating period for $f(n) = \sqrt{n}$ is shorter than that of $f(n) = 1$ case. Also, in Figure 3a there are sharp peaks observed with some kind of periodicity and more oscillations at the same period of time have been observed. This can be thought of to imply that the effects on the entanglement degree of both specific intensity-dependent coupling and the initial field photon statistics can be counterbalanced in some special cases. The case in which the intensity-dependent coupling is taken to be $f(n) = 1/\sqrt{n}$ is quite interesting where in this case the entanglement degree function oscillates around the maximum values when the time goes on. We have shown here a new phenomena the periodic oscillations occur in the presence of the intensity-dependent coupling. This difference reflects the various influences of intensity-dependent media on the interaction between atom and field. A slight change in ς_i therefore, dramatically alters the entanglement. It should be noted that at a special choice of the nonlinear intensity-dependent coupling, the situation becomes interesting, in this case, we find that the nonlinear three-level system with an initially coherent fields exhibits superstructures instead of the first-order revivals resembling those manifested by the standard three-level system. These results are similar to those obtained for an ion coupled to its motional degrees of freedom in an ion trap [44].

In the following discussion we would like to highlight another special feature of the present model. It is well-known that, in the case of large one-photon detuning, terms involving the ground-excited state coherence and excited state population can be adiabatically eliminated. The three-level system is then equivalent to an effective two-level system in which the spin associated with the ground state sublevel may be squeezed. The effect of the parameter Δ which describes the mismatch between the atomic frequency and the mean frequency of the cavity mode is considered in Figure 4. By adjusting the

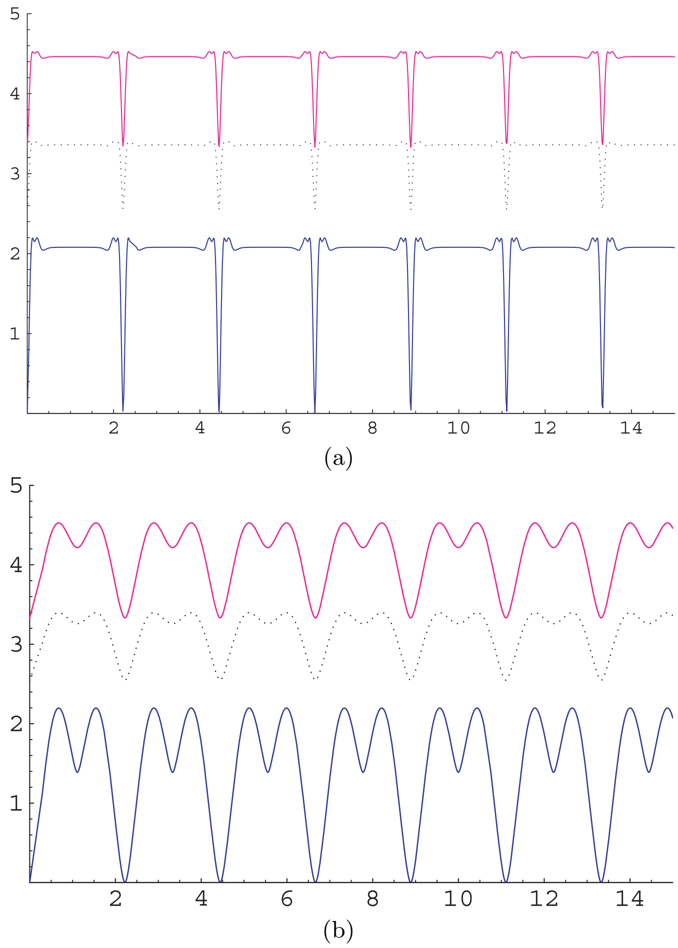


Fig. 3. The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$ as functions of the scaled time $\gamma_1 t$. The detuning parameter Δ has zero value, $\varsigma_3 = 0$ and $\varsigma_1 = 0.001$ (sold curve), $\varsigma_1 = 0.8(I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) + 2.5$, dashed-curve) and $\varsigma_1 = 0.6(I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right) + 1.5$, dotted-curve). The field initially in a coherent state with $\bar{n} = 5$, where (a) $f(n) = \sqrt{n}$, and (a) $f(n) = 1/\sqrt{n}$.

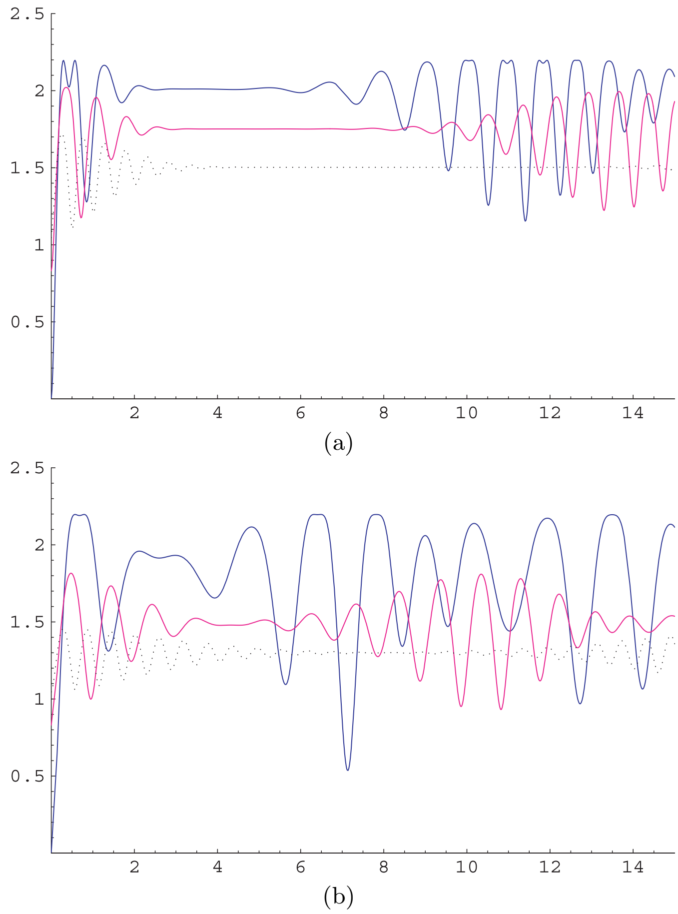


Fig. 4. The evolution of the entanglement degree $I_{\mathcal{E}_t^*\rho}\left(\rho_t^A, \rho_t^F\right)$ as functions of the scaled time $\gamma_1 t$. $\varsigma_3 = 0$ and $\varsigma_1 = 0.8$, $f(n) = 1$. The field initially in a coherent state. $\Delta = 1$ (sold curve), $\Delta = 5$ (dashed-curve) and $\Delta = 10$ (dotted-curve). (a) $\bar{n} = 5$ and (b) $\bar{n} = 1$.

other parameters as $\varsigma_1 = 0.8$ and the detuning parameter $\Delta/\gamma_1 = 1, 5, 10$. When the detuning considered we find that the situation has been changed. As we increase the value of the detuning we have more oscillations but with time of revivals prolonged, see for example Figures 4. It is also noted that the amplitude of the oscillation in this model are lesser than their counterparts for the two-level case. Finally we point out that, as we increase the value of the detuning Δ/γ_1 one can see the revival time is also prolonged, however the amplitude of fluctuations is decreasing. Detuning affects the revival time by elongating it and the maximum value of the entanglement degree becomes less and less. The same situation has been considered in Figure 4b, but with smaller mean photon number $\overline{n} = 1$. In these figures we have shown for some parameters the entanglement degree tends to zero. Of course the total atom-field state can not have its purity diminished, which means that as the field becomes more pure the atomic state must be closer to a mixed state. It should be noted that at a special choice of the atom-field coupling constants, such as $\gamma_1/\gamma_2 \gg 1$ or $\gamma_1/\gamma_2 \ll 1$, the entanglement degree exhibits superstructures instead of the first-order revivals resembling those manifested by the standard two-level system [9].

The remaining task is to identify and compare the results presented above for the entanglement degree $I_{\mathcal{E}_t^*\rho}(\rho_t^A,\ \rho_t^F)$ with another accepted entanglement measure such as the negativity. The question of the ordering of entanglement measures was raised in reference [2]. It was proved that all good asymptotic entanglement measures are either identical or fail to uniformly give consistent orderings of density matrices [45]. The best understood case, not surprisingly, is the simplest. Thus another entanglement measure to consider can be defined in terms of the negative eigenvalues of the partial transposition of the density operator, which takes the following form [6,46,47]

$$I_{\mathcal{E}_t^*\rho}(t) = 2\max\left\{0, -\lambda_{\text{neg}}^{-}\right\}, \tag{25}$$

where λ_{neg}^{-} is the sum of the negative eigenvalues of the partial transposition of the reduced atomic density matrix ρ^a, which can be obtained by tracing out the field variables. One, possibly not very surprising, principal observation is that the numerical calculations corresponding to the same parameters, which have been considered in Figures 2–4, give nearly the same behaviour. This means that both the entanglement due to the quasi-mutual entropy and negativity measures are qualitatively the same. We must stress, however, that no single measure alone is enough to quantify the entanglement in a multilevel system.

Before we conclude, it may be profitable to give a brief discussion on the experimental realization of the present model. It was reported that [48] the cavity can have a photon storage time of $T = 1$ ms (corresponding to $Q = 3 \times 10^8$). The radiative time of the Rydberg atoms with the principle quantum numbers 49, 50 and 51 is about 2×10^{-4} s. In order to realize such a scheme in laboratory experiment within microwave region, we may consider slow Rb atoms in higher Rydberg states which have life time of the order of few milliseconds [49]. These slow atoms, initially pumped to high Rydberg state, pass through a high-Q superconducting cavity of dimension of a few centimeters with a velocity of around 400 m/s [49,50]. The interaction times of atom with the cavity modes come out to be of the order of few tens of microseconds which is far less than the cavity life time. The high-Q cavities of life time of the order of millisecond are being used in recent experiments [50].

5 Summary

In this paper, we have developed a general entanglement measure technique to study the three-level quantum system interacting with a cavity-field and have shown that using quantum mutual entropy to calculate the entanglement degree open new possibilities for better understanding of multi-level systems interactions when the system initially starts from a mixed state. It is noteworthy that our approach can be applied to any three-level system interaction. As a result we obtained exact expression for the entanglement degree due to the quantum mutual entropy, highlighting the role of the intensity-dependent coupling contributions and detuning on the three-level atom interacting with a single cavity mode. Our main aspects are summarized as follows.

(i) We have derived the general analytical expressions for the entanglement degree, which are suitable for any three-level system.
(ii) For the V-type case, we have discussed how intensity-dependent coupling and detunings affect the entanglement degree. Under the proper conditions strong entanglement with a large number of photons can be easily produced in this novel scheme.
(iii) We have analyzed how the entanglement degree changes with changing the mixed state parameters. In all cases we have shown that, the entanglement degree is always decreases with increasing the mixed state parameter ς_1.
(iv) Our results indicate that it is perfectly possible to measure the entanglement degree in a general three-level system.

Finally, it should be mentioned that one can try to apply the strategy developed in this paper to the case in which atomic detuning and spontaneous emission are present.

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